Section 7.2: Trigonometric Integrals

Objective: In this lesson, you learn

☐ How to evaluate integrals involving certain products of powers of trigonometric functions.

I. Integrating trig functions: sine and cosine.

In this section, we use trigonometric identities to integrate certain combinations of trigonometric functions.

Example 1: Integrate $\int \sin^2(x)\,dx$

$\int f(x)dx$	$f(x)$	$\frac{d}{dx} f(x)$
$-\cos x + C$	$\sin x$	$\cos x$
$\sin x + C$	$\cos x$	$-\sin x$
$\ln\lvert\sec x\rvert + C$	$\tan x$	$\sec^2 x$
we will do it today	$\sec x$	$\sec x \tan x$
$-\ln\lvert\csc x\rvert + C$	$\cot x$	$-\csc^2 x$
	$\csc x$	$-\csc x \cot x$

Half-angle Identities

$$\boxed{\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x}$$

$$\int \sin^2 x\,dx = \int \frac{1}{2} - \frac{1}{2}(\cos 2x)\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin 2x\right) + C$$

$u = 2x \quad du = 2\,dx$

$\frac{du}{2} = dx$

$$\int \cos 2x\,dx$$

$$= \int \cos u \cdot \frac{du}{2} = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u$$

$$= \frac{1}{2}\sin 2x + C$$

$$\boxed{\int \sin ax\,dx = -\frac{1}{a}\cos ax + C \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + C}$$

Example 2: Integrate $\int \cot^2(3x)\,dx$

$$\boxed{\cot^2 x + 1 = \csc^2 x}$$

$$\cot^2 x = \csc^2 x - 1$$

$$\int \cot^2(3x)\,dx = \int \csc^2(3x) - 1\,dx$$

$$= \int \csc^2(3x)\,dx - \int dx$$

$$= -\frac{1}{3}\cot(3x) - x + C$$

$$\boxed{\csc^2(ax) = \frac{-1}{a}\cot(ax)}$$

Strategy for Evaluating $\int \sin^m x \, \cos^n x \, dx$

Here are the guidelines for evaluating integrals of the form $\int \sin^m x \, \cos^n x \, dx$, where $m \geq 0$ and $n \geq 0$ are integers:

a. If the power of cosine is odd ($n = 2k+1$), save one cosine factor and use $\cos^2(x) = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx = \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx$$

Then substitute $u = \sin x$.

$\boxed{\sin^2 x + \cos^2 x = 1}$

Example 3: Integrate

$$\int \cos^3 x \, \sin^2 x \, dx$$

$n = 3$ odd

$$\int \cos x \, \cos^2 x \, \sin^2 x \, dx$$

$$= \int \cos x \left(1 - \sin^2 x\right) \sin^2 x \, dx$$

$u = \sin x$ $\qquad du = \cos x \, dx$

$$= \int (1 - u^2) \, u^2 \, du$$

$$= \int u^2 - u^4 \, du = \frac{u^3}{3} - \frac{u^5}{5} + C$$

$$= \frac{\sin^3 x}{3} - \frac{\sin^5}{5} + C.$$

Example 4: Integrate

$$\int \cos^5(2x)\, dx$$

$u = 2x \to du = 2\,dx \to dx = \frac{du}{2}$

$$\int \cos^5(2x)\,dx = \frac{1}{2}\int \cos^5 u\, du$$

$n = 5$ odd

$$\frac{1}{2}\int \cos^5 u\, du = \frac{1}{2}\int \cos u \cdot \cos^4 u\, du = \frac{1}{2}\int \cos u\,(\cos^2 u)^2\, du$$

$$= \frac{1}{2}\int \cos u \cdot (1-\sin^2 u)^2\, du$$

$t = \sin u \qquad dt = \cos u\, du$

$$= \frac{1}{2}\int (1-t^2)^2\, dt = \frac{1}{2}\int 1 - 2t^2 + t^4\, dt = \frac{1}{2}\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right)$$

$$= \frac{1}{2}\left(\sin u - \frac{2}{3}\sin^3 u + \frac{\sin^5 u}{5}\right) + C$$

$$= \frac{1}{2}\left(\sin 2x - \frac{2}{3}\sin^3(2x) + \frac{\sin^5(2x)}{5}\right) + C$$

Example 5: Integrate

$n = 3$ odd

$$\int \cos^3(x)\sin(x)\, dx$$

$$\int \cos^3 x \sin x\, dx = \int \cos x\,(\cos^2 x)\sin x\, dx$$

$$= \int \cos x\,(1-\sin^2 x)\sin x\, dx$$

$u = \sin x \qquad du = \cos x\, dx$

$$= \int (1-u^2)\cdot u\, du$$

$$= \int u - u^3\, du = \frac{u^2}{2} - \frac{u^4}{4} + C$$

$$= \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C.$$

b. If the power of sine is odd ($m = 2k+1$), save one sine factor and use $\sin^2 + \cos^2 = 1$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx = \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx$$

Then substitute $u = \cos x$.

$m = 3$ odd

Example 6: Evaluate $\int \sin^3 x \; dx$.

$$\int \sin^3 x \, dx = \int \sin x \, \sin^2 x \, dx$$

$$= \int \sin x \left(1 - \cos^2 x\right) dx$$

$u = \cos x \qquad du = -\sin x \, dx$

$$= -\int 1 - u^2 \, du = -u + \frac{u^3}{3} + C$$

$$= -\cos x + \frac{\cos^3 x}{3} + C.$$

Example 7: Evaluate $\int \sin^3 x \; \cos^2 x \; dx$.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin x \left(\sin^2 x\right) \cos^2 x \, dx$$

$$= \int \sin x \left(1 - \cos^2 x\right) \cos^2 x \, dx$$

$u = \cos x \qquad du = -\sin x \, dx$

$$= -\int \left(1 - u^2\right) u^2 \quad du$$

$$= -\int u^2 - u^4 \, du$$

$$= -\frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= -\frac{(\cos x)^3}{3} + \frac{(\cos x)^5}{5} + C$$

c. If the powers of both sine and cosine are **even**, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \quad \text{and} \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

It is sometimes helpful to use the double-angle identity

$$\sin x \cos x = \frac{1}{2}\sin 2x.$$

$(ab)^2 = a^2 b^2$

Example 8: Evaluate $\int \sin^2 x \cos^2 x \, dx$.

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 \, dx$$
$$= \int \left(\frac{1}{2}\sin 2x\right)^2 dx$$
$$= \frac{1}{4}\int \sin^2 2x \, dx$$
$$= \frac{1}{4}\int \frac{1}{2}(1 - \cos 4x) \, dx$$
$$= \frac{1}{8}\left(x - \frac{1}{4}\sin 4x\right) + C$$

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}(1 - \cos 2x) \cdot \frac{1}{2}(1 + \cos 2x) \, dx$$
$$= \frac{1}{4}\int (1 - \cos 2x)(1 + \cos 2x) \, dx$$
$$= \frac{1}{4}\int 1 - \cos^2 2x \, dx$$
$$= \frac{1}{4}\int \sin^2 2x \, dx$$

$\sin^2 2x + \cos^2 2x = 1$

$a^2 - b^2 = (a-b)(a+b)$

II. Integrating other trig function: tangent, cotangent, secant, and cosecant

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

a. If the power of secant is even ($n = 2k$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^n x \, dx = \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x \, dx = \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x \, dx$$

Example 9: Evaluate $\int \tan^2 x \sec^4 x \, dx$.

$$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \underbrace{\sec^2 x \, dx}$$

$$= \int \tan^2 x (1 + \tan^2 x) \sec^2 x \, dx$$

$u = \tan x \qquad du = \sec^2 x \, dx$

$$= \int u^2 (1 + u^2) \, du$$

$$= \int u^2 + u^4 \, du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C$$

$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C.$$

b. If the power of tangent is odd ($m = 2k+1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \, \sec^n x \, dx = \int \left(\tan^2 x\right)^k \sec^{n-1} x \, \sec x \tan x \, dx = \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \, \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

odd

Example 10: Evaluate $\int \tan^3 x \, \sec^5 x \, dx$.

$$\int \tan^3 x \sec^5 x \, dx = \int \tan^2 x \sec^4 x \, \tan x \sec x \, dx$$

$$= \int (\sec^2 x - 1) \sec^4 x \, \tan x \sec x \, dx$$

$u = \sec x \qquad du = \sec x \tan x \, dx$

$$= \int (u^2 - 1) u^4 \cdot du$$

$$= \int u^6 - u^4 \, du$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5} + C$$

c. The other cases such as:

i. $\sec^n x \tan^m x$, n is odd and m is even.

ii. $\tan^m x$, m is odd and there is no $\sec x$.

may require a combination of identities, integration by parts, and a little ingenuity.

Example 11: Evaluate $\int \tan^3 x \sec^4 x\, dx$.

a. $\int \tan^3 x \sec^4 x\, dx = \int \tan^2 x \sec^3 x \sec x \tan x\, dx$

$= \int (\sec^2 x - 1) \sec^3 x \sec x \tan x\, dx$

$u = \sec x \to du = \sec x \tan x\, dx$

$= \int (u^2 - 1) u^3 \cdot du$ (H.W)

b. $\int \tan^3 x \sec^2 x \sec^2 x\, dx = \int \tan^3 x (1 + \tan^2 x) \sec^2 x\, dx$

$u = \tan x \to du = \sec^2 x\, dx$

$= \int u^3 (1 + u^2) \cdot du$ (H.W)

Example 12: Evaluate $\int \tan x\, dx$.

$a \ln x = \ln x^a$

$\int \tan x\, dx = -\int \frac{-\sin x}{\cos x}\, dx$

$= -\ln|\cos x| + C$

$= \ln|\sec x| + C$

$-\ln|\cos x|$

$= \ln\left|\frac{1}{\cos x}\right|$

$= \ln|\sec x|$

Example 12': $\int \tan^2 x\, dx = \int \sec^2 x - 1\, dx$

$\tan x - x + C$

Example 13: Evaluate $\int \tan^3 x\, dx$.

$$\int \tan^3 x\, dx = \int \tan^2 x \tan x\, dx$$
$$= \int (\sec^2 x - 1) \tan x\, dx$$
$$= \int \sec^2 x \tan x - \tan x\, dx$$
$$u = \tan x \quad du = \sec^2 x\, dx$$
$$= \int u\, du - \int \tan x\, dx$$
$$= \frac{u^2}{2} - \ln|\sec x| + C = \frac{\tan^2 x}{2} - \ln|\sec x| + C$$

Example 14: Evaluate $\int \sec x\, dx$.

$$\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)}\, dx$$
$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$
$$u = \sec x + \tan x \rightarrow du = \sec x \tan x + \sec^2 x\, dx$$
$$= \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|\sec x + \tan x| + C$$

Example 15: Evaluate $\int \sec^3 x\, dx$.

$$\int \sec^3 x \; dx = \int \sec x \sec^2 x \; dx$$

$u = \sec x \qquad du = \sec x \tan x \; dx$

$dv = \sec^2 x \, dx \qquad v = \tan x$

$$\int u \, dv = uv - \int v \, du$$

$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$- \int \sec x \, (\sec^2 x - 1) \; dx$$

$$- \int \sec^3 x - \sec x \; dx$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \; dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x| + C$$

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C.$$

To evaluate integrals of the forms $\int \sin mx \;\cos nx \; dx$, $\int \sin mx \;\sin nx \; dx$, and $\int \cos mx \;\cos nx \; dx$, use the identities

a. $\sin Ax \cos Bx = \frac{1}{2}\left[\sin{(A-B)x} + \sin{(A+B)x}\right]$

b. $\sin Ax \sin Bx = \frac{1}{2}\left[\cos{(A-B)x} - \cos{(A+B)x}\right]$

c. $\cos Ax \cos Bx = \frac{1}{2}\left[\cos{(A-B)x} + \cos{(A+B)x}\right]$

Example 16: Evaluate $\int \sin 6x \;\sin 11x \; dx$.

$$\int \sin \overset{A}{6}x \; \sin \overset{B}{11}x \; dx \overset{b}{=} \int \frac{1}{2}\left[\cos{(6-11)x} - \cos{(6+11)x}\right] dx$$

$$= \frac{1}{2}\int \cos(-5x) - \cos(17x) \; dx$$

$$= \frac{1}{2}\int \cos(5x) - \cos(17x) \; dx$$

$$= \frac{1}{2}\left(\frac{1}{5}\sin(5x) - \frac{1}{17}\sin(17x)\right) \quad \#\#$$

$f(-x) = f(x)$

since ($\cos(-x) = \cos(x)$, $\cos x$ is an even function)

$$\int \cos(-5x) \; dx$$

$$= \frac{-1}{5}\sin(-5x)$$

$$= \frac{-1}{5}(-\sin 5x)$$

$$= \frac{1}{5}\sin 5x$$

since $\sin(-x) = -\sin x$

$\sin x$ is an odd function

$f(-x) = -f(x)$